Creation of massive particles in a tunneling universe

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We examine the particle production during tunneling in quantum cosmology. We consider a minisuperspace model with a massive, conformally coupled scalar field and a uniform radiation background. In this model, we construct a semiclassical wave function describing a small recollapsing universe and a nucleated inflating universe ("tunneling from something"). We find that the quantum states of the scalar field in both the initial and the nucleated universe are close to the adiabatic vacuum, the number of created particles is small, and their backreaction on the metric is negligible. We show that the use of the semiclassical approximation is justified for this wave function. Our results imply that the creation of the universe from nothing can be understood as a limit of tunneling from a small recollapsing universe.

I. INTRODUCTION

A semiclassical picture of quantum cosmology based on the Wheeler-DeWitt equation describes tunneling from a state of vanishing size ("tunneling from nothing") to a closed inflating universe. The process of tunneling from nothing can be thought of as a limit of tunneling from a closed recollapsing universe of very small but nonzero size to an inflating universe ("tunneling from something"). This paper continues the investigation of particle creation during tunneling in quantum cosmology. Conflicting claims of excessive particle production that invalidates the semiclassical approximation [1, 2], on the one hand, and of essentially no particle content in the nucleated universe [3, 4, 5], on the other hand, have been advanced in the literature. Our intent is to resolve this long-standing controversy.

The process of tunneling from a recollapsing universe sensitively depends on the quantum

state of that universe. In the companion paper [6] we have shown that, at least in the case of a massless field, the results of Rubakov *et al.* [1, 2] should be interpreted not as an indication of a large particle production but as a consequence of an inadequate choice of the initial quantum state of the universe. A generic quantum state of the recollapsing universe will contain a superposition of various semiclassical geometries. Some geometries in the superposition do not describe a nucleated universe but rather a universe that expanded from zero size "over the barrier" without tunneling, because it contained a large number of particles. Other geometries in the superposition will describe a nucleated universe with a small number of particles. One can hope to extract well-defined particle numbers from a Wheeler-DeWitt wave function only if it describes a single semiclassical background spacetime. However, the process of tunneling amplifies the differences between branches of the wave function. A semiclassical branch of the wave function describing a universe with a large particle content may be strongly suppressed in the initial recollapsing universe but may dominate the wave function of the tunneling universe (because it avoids being exponentially suppressed during tunneling).

For a massless field, there is a well-defined vacuum and the particle production is absent [7]. The quantum state chosen by Rubakov [1] corresponds to a squeezed state with respect to that vacuum and consists of an infinite superposition of excited states with all possible energies. Each excited state is described by a branch of the wave function that has its own semiclassical geometry. As we showed in Ref. [6], one cannot neglect the difference between these semiclassical geometries if one considers the nucleated universe. We call this phenomenon "critical branching". We have shown that the "catastrophic particle creation" found by Rubakov *et al.* is in fact a sign of critical branching that happens with their choice of the initial state. If one chooses the initial state to be the vacuum, or any excited state that is an eigenstate of the particle occupation numbers, then there is no branching, and the wave function describes a single semiclassical geometry.

The purpose of the present paper is to extend the analysis of Ref. [6] to the case of a massive field. A nonzero mass m introduces important qualitative differences into the problem. First, a massive scalar field is nontrivially coupled to the time-dependent metric, and there is necessarily some particle production. Even if one imagines starting the recollapsing universe in a vacuum state at early times, the field will be in a superposition of states with all possible occupation numbers at other times. Additional creation of excited states may

occur as a result of tunneling. Thus, it is not clear *a priori* that critical branching can be avoided for any choice of the initial state. Moreover, the definition of the vacuum state for a massive field in an expanding universe is notoriously ambiguous; as a result, the number of particles is also subject to ambiguity. In this paper we give a detailed analysis of these issues.

The approach taken by Rubakov *et al.* [1, 2] was to define the vacuum state by diagonalizing the scalar field Hamiltonian at a moment of time. However, this procedure is problematic since it is known to give unphysical results for the particle density in some cases [8]. A well-motivated definition of vacuum in a slowly changing background geometry has been developed by Parker and Fulling; it is the so-called adiabatic vacuum [7, 9]. An adiabatic vacuum of k-th order is defined using a truncated asymptotic series obtained from the WKB expansion. It has some dependence on the moment of time when it is defined and on the order k at which the series is truncated. The resulting uncertainty is of order of the k-th power of the adiabatic parameter. On the other hand, it follows from our analysis in [6] that an admissible state of the universe must be defined with an exponential accuracy to avoid the "critical branching". We need to specify a vacuum state with a higher accuracy than the definition of the adiabatic vacuum allows.

Our main result in this paper is a prescription for constructing a well-behaved semiclassical state of the universe which does not exhibit critical branching. We show that this state always exists and coincides with the adiabatic vacuum within the accuracy to which the latter can be defined. It also coincides with the canonically defined vacuum in the massless case.

The paper is organized as follows. In Sec. II we construct a family of well-behaved semiclassical wave functions of the universe. We give a prescription to select a unique quantum state that we call the "Gaussian vacuum". In this state the backreaction of produced particles is negligible and there is no branching. We obtain approximate expressions for the wave function of this state. In Sec. III we give a particle interpretation of the Gaussian vacuum state. We show that the Gaussian vacuum is indistinguishable from an adiabatic vacuum state to all allowed orders of the adiabatic expansion. In Sec. IV we discuss our results as well as the issues raised by Rubakov *et al.* We also compare our results to those of Bouhmadi-López *et al.* [10] who addressed some of the same issues. In the Appendices we give technical details of the calculations and, in particular, check the validity of our

approximations.

II. THE GAUSSIAN VACUUM STATE

We consider a homogeneous closed FRW metric and an inhomogeneous massive conformally coupled scalar field. The classical action of the system is

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi} - \frac{3}{8\pi}H^2 \right.$$
$$\left. +\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{12}R\phi^2\right\}. \tag{1}$$

Here R is the scalar curvature and the parameter H represents the cosmological constant. We use the Planck units, $G = \hbar = c = 1$. We assume $H \ll 1$ and $m \ll 1$ as a typical physically motivated case.

We expand the inhomogeneous scalar field in spherical harmonics on the 3-sphere,

$$\phi(x,t) = \frac{\pi\sqrt{2}}{a(t)}\sum_{n,l,p}\chi_{nlp}(t)Q^n_{lp}\left(\mathbf{x}\right). \tag{2}$$

A rescaling by the factor $a(t)$ is done for convenience. Below, only the index n will enter the equations, and we suppress the indices l, p of the modes $\chi_{nlp}(t)$. The summation over degenerate indices l, p spans $l = 0, ..., n-1$ and $p = -l, ..., l$ and introduces an extra factor n^2 which we shall insert in explicit calculations below.

The wave function of the universe depends on all modes χ_n of the scalar field, $\Psi = \Psi\left(a, \{\chi_n\}\right)$. The Wheeler-DeWitt equation, after appropriate rescalings [3], takes the form

$$\left[\hbar^2\frac{\partial^2}{\partial a^2} - a^2 + H^2a^4\right]\Psi$$
$$+\sum_n\left[-\hbar^2\frac{\partial^2}{\partial\chi_n^2} + n^2\chi_n^2 + m^2a^2\chi_n^2\right]\Psi = 0. \tag{3}$$

Here and below we explicitly write the Planck constant $\hbar \equiv 1$ only as a formal bookkeeping parameter, to clarify the use of the WKB approximation.

In addition to the scalar field ϕ we now include a small amount of homogeneous radiation with energy density

$$\rho_r = a^{-4}\varepsilon_r, \tag{4}$$

where $\varepsilon_r > 0$ is a constant parameter. Then the Wheeler-DeWitt equation is modified,

$$\left[\frac{\hbar^2\partial^2}{\partial a^2} - V(a) + \varepsilon_r + \sum_n\left[-\frac{\hbar^2\partial^2}{\partial\chi_n^2} + \omega_n^2\chi_n^2\right]\right]\Psi = 0, \tag{5}$$

where we have defined

$$V(a) \equiv a^2 - H^2 a^4, \quad (6)$$

$$\omega_n(a) \equiv \sqrt{n^2 + m^2 a^2}. \quad (7)$$

If restricted to the coordinate a, Eq. (5) is a stationary Schrödinger equation for a particle in a potential.

A. Gaussian solutions of the WDW equation

In the companion paper [6] we have used the method of Refs. [11, 12, 13, 14] to find an approximate solution of Eq. (5). The solution may be found as a linear combination of Gaussian terms of the form

$$\Psi\left(a, \{\chi_n\}\right) = \exp\left[-\frac{S(a)}{\hbar} - \frac{1}{2\hbar}\sum_n S_n(a)\chi_n^2\right], \quad (8)$$

where $S(a)$ and $S_n(a)$ are functions to be determined. The functions $S_n(a)$ must satisfy the condition that we shall call the "regularity condition",

$$0 < Re\, S_n(a) < +\infty \text{ for all } a,\, n. \quad (9)$$

With this condition, the Gaussian wave function of Eq. (8) is well-defined everywhere and quickly decays at large χ_n. In Ref. [6] we have shown that a violation of the regularity condition indicates a splitting of the wave function into decoherent branches with macroscopically different semiclassical geometries and different particle contents (critical branching). A wave function of the universe can be consistently interpreted in terms of a classical spacetime with a quantum field only if a single underlying semiclassical geometry is present. Each term of the form (8) will describe a single semiclassical geometry if the regularity condition holds for its function $S_n(a)$. Then the branches of the wave function corresponding to each such term will describe independent, decoherent semiclassical universes. Therefore we may impose the regularity condition on all terms of the form of Eq. (8) that comprise the wave function of the universe.

We now substitute the ansatz of Eq. (8) into Eq. (5) and obtain

$$
\begin{aligned}
(S')^2 - V(a) - \hbar S'' + \hbar \sum_n S_n &= 0, && (10)\\
\left[S' S_n' - S_n^2 + [\omega_n(a)]^2 - \frac{\hbar}{2} S_n'' \right] \chi_n^2 & \\
+ \frac{1}{4} (S_n')^2 \chi_n^4 &= 0. && (11)
\end{aligned}
$$

Our approximation consists of disregarding the terms of order $O(\hbar)$ and $O(\chi_n^4)$; the applicability of this approximation is analyzed in Appendix B. We then have the following equations for the functions $S(a)$ and $S_n(a)$,

$$
\begin{aligned}
(S')^2 - V(a) + \varepsilon_r &= 0, && (12)\\
S' S_n' - S_n^2 + \omega_n^2 &= 0. && (13)
\end{aligned}
$$

It is clear that $\exp(-S/\hbar)$ is a standard WKB wave function for a particle in a potential $V(a)$ with energy ε_r. For $H^2 \varepsilon_r \ll 1$ the turning points $a_{1,2}$ are approximated by

$$
\begin{aligned}
a_1^2 &\equiv \frac{1 - \sqrt{1 - 4\varepsilon_r H^2}}{2H^2} \approx \varepsilon_r, && (14)\\
a_2^2 &\equiv \frac{1 + \sqrt{1 - 4\varepsilon_r H^2}}{2H^2} \approx H^{-2}. && (15)
\end{aligned}
$$

If $4H^2 \varepsilon_r < 1$ then $a_1 \neq a_2$ are real and there exist two classically allowed regions and a classically forbidden region (see Fig. 1). The physical picture of the resulting cosmology is the following. A small closed universe filled with radiation of energy density $\varepsilon_r a^{-4}$ expands until the maximum scale factor a_1 and then recollapses to $a = 0$. An expanding universe is created with a scale factor a_2 by tunneling through the potential barrier. If the WKB approximation is valid, then the scale factor a is a semiclassical variable and any monotonic function of a can be used as a time coordinate in the two classically allowed Lorentzian regions. The turning point $a = a_2$ corresponds to the beginning of time in the nucleated expanding universe; the Euclidean region $a_1 < a < a_2$ does not correspond to a classical universe. If $H = 0$, the second turning point is absent (formally $a_2 \to +\infty$) and there is only one Lorentzian region $0 < a < a_1$.

We may rewrite Eqs. (12)-(13) as

$$
\begin{aligned}
S(a) &= \pm \int^a \sqrt{V(a) - \varepsilon_r}\, da, && (16)\\
\pm \sqrt{V(a) - \varepsilon_r} S_n' &= S_n^2 - \omega_n^2. && (17)
\end{aligned}
$$

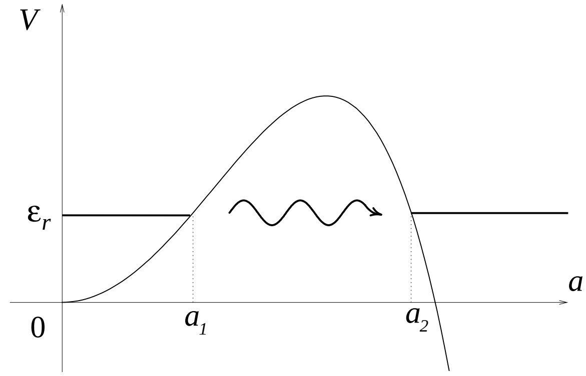

Figure 1: “Tunneling from something”: creation of an expanding universe by tunneling.

In the two Lorentzian regions $[0 < a < a_1$ and $a > a_2]$ the square root in Eq. (16) assumes imaginary values (with a positive imaginary part) and the upper sign denotes a wave $\exp[-S(a)]$ traveling to the right, with

$$S(a) = i \int^{a} \sqrt{\varepsilon_r - V(a)}\, da. \tag{18}$$

In the Euclidean region $[a_1 < a < a_2]$ the square root in Eq. (16) is real and the upper sign denotes an exponentially decaying mode $\exp[-S(a)]$ with

$$S(a) = \int^{a} \sqrt{V(a) - \varepsilon_r}\, da. \tag{19}$$

We shall denote by $S_n(a)$ and $S_n^-(a)$ the solutions of Eq. (17) with the upper and the lower signs respectively, omitting the $^+$ subscript in $S_n^+(a)$ for brevity. If $S(a)$ is given by Eqs. (18)-(19), then a linear combination

$$\begin{aligned} \Psi\left(a, \{\chi_n\}\right) &= C_+ \exp\left[-\frac{S(a)}{\hbar} - \sum_n S_n(a) \frac{\chi_n^2}{2\hbar} \right] \\ &+ C_- \exp\left[\frac{S(a)}{\hbar} - \sum_n S_n^-(a) \frac{\chi_n^2}{2\hbar} \right] \end{aligned} \tag{20}$$

is an approximate solution of the Wheeler-DeWitt equation. The coefficients $C_\pm$ and the solutions S_n, S_n^- must be chosen separately in each of the three physical domains (two Lorentzian regions and one Euclidean region). To match these solutions across the turning points, we need to use the matching conditions of Ref. [4]. These conditions state, in particular, that the value of S_n must be continuous across the turning points $a_{1,2}$ and moreover

$$\lim_{a \to a_1 - 0} S_n(a) = \lim_{a \to a_1 - 0} S_n^-(a) = \lim_{a \to a_1 + 0} S_n(a). \tag{21}$$

A similar matching condition must be satisfied at the second turning point a_2,

$$\lim_{a\to a_2-0} S_n(a) = \lim_{a\to a_2-0} S_n^-(a) = \lim_{a\to a_2+0} S_n(a). \tag{22}$$

Due to the tunneling boundary condition at $a \to +\infty$ (no wave traveling to the left) the coefficient C_- for the region $a > a_2$ must vanish and we do not need the branch $S_n^-(a)$ in that region. In addition to Eqs. (21)-(22) we require that the regularity condition [Eq. (9)] holds for $S_n(a)$, $S_n^-(a)$. We shall refer to such solutions of Eq. (17) as "regular".

Note that the function $S_n^-(a)$ does not have to be continuous at $a = a_1$. This is because the branch with the lower signs [the second term in Eq. (20)] is exponentially small at $a = a_1$ compared with the other branch and cannot be kept in the WKB approximation.

B. Construction of regular Gaussian solutions

Our task here is to obtain an explicit solution of

$$\sqrt{V(a) - \varepsilon_r}\frac{dS_n}{da} = S_n^2 - \omega_n^2, \tag{23}$$

subject to the regularity condition [Eq. (9)],

$$0 < Re\, S_n(a) < \infty \text{ for all } a. \tag{24}$$

Here $S_n(a)$ is in general a complex function. Because of matching conditions, $S_n(a)$ must be continuous across the turning points $a = a_{1,2}$. Equation (23) is identical to Eq. (17); the equation for $S_n^-(a)$ can be obtained by changing the sign of the derivative d/da in Eq. (23).

Following Refs. [11, 12, 13], we introduce the semiclassical time variables: t in the classically allowed regions and τ in the under-barrier region,

$$\frac{da}{dt} \equiv \sqrt{\varepsilon_r - V(a)}, \quad a < a_1 \text{ or } a > a_2, \tag{25}$$

$$\frac{da}{d\tau} \equiv \sqrt{V(a) - \varepsilon_r}, \quad a_1 < a < a_2. \tag{26}$$

The variable t is the conformal time in a Lorentzian universe, while τ is the Euclidean conformal time. Below we shall be using the variables a, t and τ interchangeably in the appropriate ranges of a. Then Eq. (23) becomes

$$i\frac{dS_n}{dt} = S_n^2 - \omega_n^2, \quad a < a_1 \text{ or } a > a_2, \tag{27}$$

$$\frac{dS_n}{d\tau} = S_n^2 - \omega_n^2, \quad a_1 < a < a_2. \tag{28}$$

The equations for the other branch S_n^- of Eq. (17) are obtained by reversing the signs of the time derivatives. The branch S_n corresponds to the wave traveling to the right in the classically allowed regions and to the decaying underbarrier branch in the Euclidean region.

To analyze the existence of solutions and to obtain approximations, it is convenient to transform the function $S_n(a)$ and introduce an auxiliary function

$$\zeta_n(a) \equiv \frac{S_n - \omega_n}{S_n + \omega_n}. \tag{29}$$

We imply that the pair of functions $S_n^\pm(a)$ is transformed into a pair $\zeta_n^\pm(a)$ but will often omit the superscripts $^\pm$ for brevity. The simple transformation of Eq. (29) allows to use the functions $S_n^\pm$ and $\zeta_n^\pm$ interchangeably. The function $\zeta_n(t)$ can be interpreted as the "instantaneous squeezing parameter" [6] describing a state of the mode χ_n that is related to the vacuum of the instantaneous diagonalization picture by a time-dependent Bogolyubov transformation. However, for our purposes it is enough to regard Eq. (29) as a formal transformation that simplifies calculations.

The regularity condition [Eq. (9)] is now equivalent to

$$|\zeta_n(a)| < 1. \tag{30}$$

This condition is violated by $|\zeta_n| = 1$ when $Re\, S_n(a) = 0$ or by $\zeta_n = 1$ when $|S_n(a)| \to \infty$. From Eq. (29) and the matching conditions for $S_n(a)$ it follows that the (complex) function $\zeta_n(a)$ must be continuous across the turning points $a_{1,2}$. From Eqs. (27), (28) we obtain the equations for $\zeta_n(a)$ in each region in terms of the conformal time variables,

$$\frac{d}{dt}\zeta_n = 2i\omega_n\zeta_n - \frac{1}{2\omega_n}\frac{d\omega_n}{dt}(1-\zeta_n^2), \tag{31}$$

$$\frac{d}{d\tau}\zeta_n = 2\omega_n\zeta_n - \frac{1}{2\omega_n}\frac{d\omega_n}{d\tau}(1-\zeta_n^2). \tag{32}$$

Our goal now is to obtain a solution for $\zeta_n(a)$ satisfying the regularity condition.

In Appendix A we derive some technical results concerning regular solutions of Eqs. (31)-(32). In Appendix A 1 it is shown that a solution $\zeta_n(a)$ is regular if $|\zeta_n(a_2)| < 1$, while the solution $\zeta_n^-(a)$ only needs to be regular at $a = a_1$ to be regular everywhere. From this it follows that regular solutions of Eq. (17) always exist. For instance, one could numerically integrate Eq. (17) with the upper sign, starting at $a = a_2$ with any value $S_n(a_2)$ such that $0 < Re\, S_n(a_2) < +\infty$, and obtain a regular solution at $a > a_2$ and at $a < a_2$.

To build a perturbative expansion, Eqs. (31)-(32) can be rewritten as integral equations. For instance, the function $\zeta_n(\tau)$ in the Euclidean region, with an arbitrary boundary value $\zeta_n(\tau_2)$, satisfies

$$\zeta_n(\tau) = \zeta_n(\tau_2)\exp\left[-2\int_\tau^{\tau_2}\omega_n d\tau\right] + \int_\tau^{\tau_2}\frac{\dot{\omega}_n}{2\omega_n}\left[1-\zeta_n^2(\tau')\right]\exp\left[-2\int_\tau^{\tau'}\omega_n d\tau\right]d\tau'. \tag{33}$$

This equation can be solved iteratively starting with $\zeta_n(\tau) \equiv 0$. A similar calculation is performed for $\zeta_n(t)$ in the Lorentzian regions. In Appendix A 2 we prove that the sequence of iterations always converges.

An important case is an adiabatic (slow) expansion of the universe. The adiabaticity condition is

$$\left|\frac{1}{\omega_n^2}\frac{d\omega_n}{dt}\right| = \frac{m^2 a\sqrt{|\varepsilon_r - V(a)|}}{(n^2+m^2a^2)^{3/2}} \ll 1. \tag{34}$$

Analyzing Eq. (34) with $V(a)$ given by Eq. (6), we find that it holds if $n \gg m/H$ or if $m \gg H$. The only case when the ratio of Eq. (34) is of order 1 is when $n \sim 1$ and $m \sim H$.

In Appendix A 2 we also show under assumption of adiabaticity that there exist solutions for which $|\zeta_n(a)|$ is always small, of order $|\dot{\omega}_n/\omega_n^2| \ll 1$, and obtain the following approximations to general solutions [see also Eq. (A14)],

$$\begin{aligned}\zeta_n(\tau) &= \zeta_n(\tau_2)\exp\left[-2\int_\tau^{\tau_2}\omega_n d\tau\right] \\ &+ \int_\tau^{\tau_2}\exp\left[-2\int_\tau^{\tau'}\omega_n d\tau\right]\frac{\dot{\omega}_n}{2\omega_n}(\tau')d\tau',\end{aligned} \tag{35}$$

$$\begin{aligned}\zeta_n^-(\tau) &= \zeta_n^-(\tau_1)\exp\left[-2\int_{\tau_1}^{\tau}\omega_n d\tau\right] \\ &- \int_{\tau_1}^{\tau}\exp\left[-2\int_{\tau'}^{\tau}\omega_n d\tau\right]\frac{\dot{\omega}_n}{2\omega_n}(\tau')d\tau'.\end{aligned} \tag{36}$$

Here the given boundary values $\zeta_n(\tau_2)$ and $\zeta_n^-(\tau_1)$ must be sufficiently small, but are otherwise arbitrary. Similar expressions hold for the functions $\zeta_n(t)$ and $\zeta_n^-(t)$ in Lorentzian regions.

Using the above approximations for ζ_n and ζ_n^-, we find $S_n^\pm$ and find the Gaussian wave function of Eq. (20). Calculations verifying the validity of the Gaussian approximation in the neighborhood of $\chi_n = 0$ and the validity of the WKB approximation for a regular Gaussian

solution are given in Appendix B. We conclude that the regular Gaussian wave function is a valid approximation to an exact solution of the Wheeler-DeWitt equation.

C. Non-uniqueness of regular wave functions

The solutions ζ_n, ζ_n^- of Eqs. (35)-(36) depend on the arbitrary boundary values $\zeta_n(\tau_2)$ and $\zeta_n^-(\tau_1)$ respectively. The choice of these boundary values is *a priori* constrained only by regularity and by matching conditions. The matching condition $\zeta_n(\tau_2) = \zeta_n^-(\tau_2)$ leaves one free parameter in the resulting wave function.

However, this remaining freedom of choosing a regular solution of Eq. (17) does not lead to an appreciable variation in the resulting functions $S_n^{\pm}(a)$ in the Lorentzian regions. The properties of Eq. (17) in the Euclidean region $a_1 < a < a_2$ are such that, as long as the regularity condition holds, $S_n^-(a_2)$ is almost insensitive to the initial value $S_n^-(a_1)$, and in turn, the value $S_n(a_1)$ is almost insensitive to $S_n(a_2)$. More precisely, the initial value $S_n^-(a_1)$ may vary throughout a wide range but the final value $S_n^-(a_2)$ is constrained to be within an exponentially small interval.

This can be seen directly from Eqs. (35)-(36). For instance, the influence of the boundary condition $\zeta_n(\tau_2)$ on the value $\zeta_n(\tau_1)$ is suppressed by an exponentially small factor

$$\exp\left[-2\int_{\tau_1}^{\tau_2}\omega_n(\tau)d\tau\right] \equiv \exp\left[-2\theta_b\right]. \tag{37}$$

Here θ_b is in the notation of Appendix A 2 [see Eq. (A5)]. The quantity θ_b is never small since $a \sim H^{-1}$, $V(a) \sim H^{-2}$ and

$$\theta_b = \int_{a_1}^{a_2}\frac{\omega_n(a)da}{\sqrt{V(a)-\varepsilon_r}} \sim \sqrt{n^2+\frac{m^2}{H^2}}. \tag{38}$$

We find that $\theta_b \gg 1$ when $n \gg 1$ or $m \gg H$.

Since $S_n(a_2) = S_n^-(a_2)$, the value $S_n(a_1)$ is also constrained to a narrow interval. Small changes of boundary values of the functions $S_n(a)$, $S_n^-(a)$ at the turning points $a_{1,2}$ lead to small changes of their values in the Lorentzian regions. Therefore, different choices of the free parameter $S_n^-(a_1)$ yield only exponentially small changes in the resulting solutions within the Lorentzian universes. The width of the possible range of solutions within a Lorentzian region is

$$\left|\frac{\Delta S_n(t)}{S_n(t)}\right| \sim \exp(-2\theta_b). \tag{39}$$

A physically interesting simple case is that of a single closed universe, $H = 0$. In that case there is only one Lorentzian region $0 < a < a_1$ and the wave function in the Euclidean region has only one branch, $S_n(\tau)$. In the limit $H \to 0$, we find $\theta_b \to +\infty$. Therefore the influence of the boundary condition $S_n(a_2)$ at $a_2 \to +\infty$ is completely suppressed and the value $S_n(a_1)$ is uniquely determined by regularity. Similarly, the regular wave function is unique in the limiting case $\varepsilon_r \to 0$, also because $\theta_b \to +\infty$ [the integral in Eq. (38) diverges near $a = 0$]. [This conclusion does not depend on the adiabaticity condition implicit in Eq. (39); in Appendix A 6 we prove the uniqueness of the regular solution under much weaker assumptions.] However, in the general case ($H \neq 0$ and $\varepsilon_r \neq 0$) the wave function is not uniquely fixed by the regularity conditions alone.

D. Prescription for a unique Gaussian vacuum

In the previous section we have found a one-parametric family of well-behaved Gaussian wave functions that can serve as vacuum states. We may impose an additional constraint on the wave function to specify it uniquely.

The extra condition is that the branch $S_n^-(a)$ must be continuous across the first turning point a_1. Together with other matching conditions this gives

$$S_n^-(a_1) = S_n(a_1). \tag{40}$$

In Appendix A 5 we show that this condition is always satisfied by a unique pair of regular solutions $S_n(a)$, $S_n^-(a)$. This proves the existence and uniqueness of the selected state. We shall call this state a "Gaussian vacuum".

We view the additional constraint of Eq. (40) as a prescription for defining a unique regular vacuum state. The wave function resulting from this prescription has the advantage that it agrees with the vacuum obtained in Ref. [6] for the massless scalar field. All other regular solutions obtained in Sec. II C differ from the Gaussian vacuum by an exponentially small amount (in Lorentzian regions).

Note that the Gaussian vacuum prescription is not local, since it requires knowledge of the behavior of the of the frequency $\omega_n(a)$ and of the potential $V(a)$ everywhere under the barrier.

III. INTERPRETATION OF THE GAUSSIAN VACUUM AND PARTICLE PRODUCTION

A semiclassical Gaussian wave function is interpreted by making a transition from the Schrödinger picture of minisuperspace to a QFT in curved spacetime [11, 12, 13, 15]. Naturally, this can be done only in a Lorentzian region where a semiclassical spacetime exists. One introduces the conformal time t via Eq. (25). The scalar field χ is canonically quantized using the standard creation and annihilation operators $\mathbf{a}_n$, $\mathbf{a}_n^\dagger$ for the modes χ_n [here the extra indices l, p of the modes are again suppressed for brevity, cf. Eq. (2)]. The amplitude χ_n of the n-th mode is promoted to an operator $\hat{\chi}_n$ and decomposed into creation and annihilation operators,

$$\hat{\chi}_n(t) = \mathbf{a}_n \nu_n(t) + \mathbf{a}_n^\dagger \nu_n^*(t). \tag{41}$$

A vacuum state is defined by a particular choice of the mode functions $\nu_n(t)$. If the Gaussian wave function is given by Eq. (8) with a certain set of $S_n(a)$ for all n, then the corresponding mode functions ν_n are found from

$$S_n(t) = \frac{i}{\nu_n}\frac{d\nu_n}{dt}, \quad 0 < a < a_1 \,\mathrm{or}\, a > a_2. \tag{42}$$

The interpretation of the Gaussian wave function [Eq. (8)] in the Lorentzian regions is that the quantized scalar field is in the vacuum state defined by the above mode functions $\nu_n(t)$.

The mode functions ν_n are determined by Eq. (42) up to an arbitrary constant factor. It follows from Eq. (27) that $\nu_n(t)$ satisfy

$$\frac{d^2\nu_n}{dt^2} + \omega_n^2 \nu_n = 0. \tag{43}$$

This is the usual equation for the mode functions of the n-th mode of a (rescaled) conformally coupled scalar field. The mode functions may be normalized by the Wronskian condition

$$\dot{\nu}_n^* \nu_n - \dot{\nu}_n \nu_n^* = i. \tag{44}$$

A. The Gaussian state and the adiabatic vacuum

In the companion paper [6] we have shown that a Gaussian wave function satisfying the regularity condition can be interpreted as a single semiclassical spacetime with a quantized field in a certain vacuum state. In Sec. II D we have defined a unique vacuum state (the

"Gaussian vacuum") that satisfies the regularity condition. Now we need to compare this Gaussian vacuum with a physically motivated vacuum in our Lorentzian universes.

Neither the spacetime of the recollapsing universe nor that of the nucleated universe possess a well-defined asymptotically static regime. This does not permit one to define unique "in" or "out" vacuum states of the QFT in these spacetimes. However, one can define an approximate "adiabatic vacuum" of the QFT in a nonstationary FRW spacetime [9] if the adiabatic approximation is valid for the mode functions of the field. We now show that the adiabatic vacuum in fact coincides with the Gaussian vacuum state we defined (up to a small correction). The difference between the two vacua is in any case smaller than the uncertainty inherent in the definition of the adiabatic vacuum.

We assume that the adiabaticity condition holds; its precise formulation is

$$\frac{1}{\omega^2}\left|\frac{d\omega_n}{dt}\right| \ll 1 \tag{45}$$

[see also Eq. (A6)]. In that case one can use at least a few terms of the WKB expansion for Eq. (43). To obtain the WKB expansion, one can introduce a formal parameter λ, define $\omega_n(t) \equiv \omega_n(\lambda t)$ and use the ansatz

$$\nu_n(t) = \frac{1}{\sqrt{W(t)}} \exp\left[-i\int^t W(t)dt\right]. \tag{46}$$

The WKB function $W(t)$ is found as an asymptotic series in λ^2. The first few terms are

$$W = \omega_n - \lambda^2\left(\frac{\ddot{\omega}_n}{4\omega_n^2} - \frac{3\dot{\omega}_n^2}{8\omega_n^3}\right) + ... \tag{47}$$

Each derivative of ω_n adds a power of λ; at the end we put $\lambda = 1$. The true small parameter of the expansion is the slowness of change of $\omega_n(t)$, formalized through $|\dot{\omega}_n| \ll \omega_n^2$ and analogous conditions on higher time derivatives.

The definition of an adiabatic vacuum depends on the chosen adiabatic order k and on an arbitrary fiducial time t_0. Let the function $W^{(k)}(t)$ give the WKB series [Eq. (47)] truncated up to terms $O\left(\lambda^{2k}\right)$. The mode function $\nu_n(t)$ describing the adiabatic vacuum are required to coincide with the WKB solution of order k at $t = t_0$. The condition for this is

$$\frac{\dot{\nu}_n}{\nu_n} + iW^{(k)} + \frac{1}{2}\frac{d}{dt}\ln W^{(k)}\bigg|_{t=t_0} = 0. \tag{48}$$

In an adiabatically changing background, the adiabatic vacuum of order k gives an apparent particle creation rate of order $k+1$ in the adiabatic parameter. For a given metric the WKB

solution is usable only until a certain finite order $k_{\max}$ after which the WKB series starts to diverge.

The condition of Eq. (48) fixes the value $S_n(t_0)$. [For comparison, the instantaneous diagonalization approach sets $S_n(t_0) = \omega_n(t_0)$.] In Appendix A 3 we show that the asymptotic series obtained for $S_n(t)$ in the WKB approximation is the same (to all orders) as the asymptotic series for $S_n(t)$ obtained from the Gaussian vacuum. Therefore, the Gaussian vacuum coincides with the adiabatic vacuum within the accuracy of its definition.

This coincidence is not specific to the Gaussian vacuum. Any other regular Gaussian wave function as found in Sec. II C will give a value $S_n(t_0)$ different by exponentially small terms of order $\exp(-2\theta_b)$. The WKB series contains terms of order $[\dot{\omega}_n/\omega_n^2]^k$ and is insensitive to exponentially small contributions. In other words, the definition of an adiabatic vacuum of order k at time t_0 contains inherent uncertainties of much larger magnitude. We conclude that the regularity of the wave function specifies a quantum state of the universe that is indistinguishable from an adiabatic vacuum of any applicable order.

B. Quantum state of the nucleated universe

Previous work suggests that in the case of tunneling from nothing ($\varepsilon_r = 0$) the scalar field in the nucleated universe should be in the Bunch-Davies (BD) vacuum state. Now we can consider the case of tunneling from something ($\varepsilon_r \neq 0$) and compare the Gaussian vacuum state in the asymptotic region $a \to +\infty$ with the BD vacuum.

In the Schrödinger picture of minisuperspace, the mode functions $\nu_n(t)$ of a vacuum are determined by the solutions $S_n(t)$ [Eq. (42)]. Suppose that another vacuum state is determined by another set of solutions $\tilde{S}_n(t)$ and the corresponding mode functions $\tilde{\nu}_n(t)$. The two vacua are related by a Bogolyubov transformation,

$$\tilde{\nu}_n(t) = \alpha\nu_n(t) + \beta\nu_n^*(t). \tag{49}$$

We can express the Bogolyubov coefficients α_n, β_n directly through the functions $S_n(t)$ and $\tilde{S}_n(t)$, as follows. If both sets of mode functions are normalized [Eq. (44)], then α_n and β_n satisfy $|\alpha_n|^2 - |\beta_n|^2 = 1$. We can select any value of t and express $\tilde{\nu}_n(t)$ and $\dot{\tilde{\nu}}_n(t)$ according

to Eq. (49). The solution is

$$\alpha_n = \dot{\nu}_n^* \tilde{\nu}_n - \nu_n^* \dot{\tilde{\nu}}_n, \tag{50}$$

$$\beta_n = -\dot{\nu}_n \tilde{\nu}_n + \nu_n \dot{\tilde{\nu}}_n. \tag{51}$$

Then the physically measurable quantity $|\beta_n|^2$ that gives the mean occupation number in the mode χ_n is found (again, independently of a fixed value of t) as

$$|\beta_n|^2 = \frac{\left|\tilde{S}_n(t) - S_n(t)\right|^2}{4Re\ \tilde{S}_n(t) Re\ S_n(t)}. \tag{52}$$

In the case $\varepsilon_r = 0$ the regular solutions $S_n^{\pm}(a)$ of Eq. (17) are unique and are known to correspond to the de Sitter-invariant BD vacuum [4, 5]. The difference between the cases $\varepsilon_r = 0$ and $\varepsilon_r \neq 0$ can be seen from Eqs. (29), (35): the neighborhood of the first turning point $\tau = \tau_1$ contributes an exponentially small amount to $S_n^-(\tau_2)$. Therefore the regular solution in the $\varepsilon_r \neq 0$ case [denoted for now $\tilde{S}_n(a)$] may differ from the BD solution $S_n(a)$ only by an exponentially small contribution of order $\exp(-2\theta_b)$. From Eq. (52) we find that the Bogolyubov transformation relating the two vacua is exponentially close to the identity transformation ($\alpha_n = 1$, $\beta_n = 0$).

We conclude that the Gaussian vacuum state derived from the regularity of the wave function contains an exponentially small number of particles from the point of view of the BD vacuum.

IV. DISCUSSION

We have shown that a minisuperspace model of quantum cosmology with a conformally coupled massive scalar field admits a vacuum state described by a well-behaved Gaussian wave function (the "Gaussian vacuum"). Such a vacuum state is provided by a regular Gaussian solution of the form of Eq. (8) and represents a recollapsing universe and an expanding universe that is nucleated by quantum tunneling. The obtained wave function describes a single semiclassical geometry throughout the Lorentzian and Euclidean regions and does not exhibit any "branching" into different semiclassical geometries. We have checked the consistency of the WKB approximation, of the Gaussian approximation, and of neglecting the backreaction of the scalar field perturbations. We have also shown that the Gaussian

vacuum describes both the recollapsing and the expanding universe in quantum states that are indistinguishable from an adiabatic vacuum of any meaningful order.

Unlike the definition of an adiabatic vacuum, our prescription for the Gaussian vacuum is not local. In other words, we cannot specify our preferred quantum state at some value of $a < a_1$ without a full knowledge of the shape of the barrier $V(a)$ and of the function $\omega_n(a)$ under the barrier. Nevertheless we believe that our choice of the vacuum state is adequate, for the following reasons. (i) Our prescription is in the spirit of the definition of an adiabatic vacuum which was designed to minimize the apparent particle production. (ii) The Gaussian vacuum coincides with any adiabatic vacuum within the inherent uncertainty of the latter. (iii) In the massless case, where a well-motivated independent definition of the vacuum is available, our prescription selects the correct vacuum state. (iv) Finally, the Gaussian vacuum represents a fixed semiclassical background geometry. Any excited states built from the Gaussian vacuum using a finite number of creation operators are also well behaved and represent single geometries. If we wished to consider tunneling from a different state of the recollapsing universe, e.g., from a squeezed state, it would be more illuminating to represent that state as an infinite superposition of excited states built over the Gaussian vacuum, with each branch representing a single semiclassical geometry. The physical interpretation of the QFT in resulting spacetimes would be unambiguous. It would be interesting if the Gaussian vacuum (or some state exponentially close to it) could be specified by a set of local conditions, but at present we are unable to provide such a specification.

Our construction of the Gaussian vacuum is general and not specific to the particular physical system we considered. The recent work of Bouhmadi-López *et al.* [10] discusses the quantum cosmology of a FRW universe filled with massless radiation field and a conformally coupled massive scalar field, in the presence of a positive or a negative cosmological constant. They have used the regularity conditions to constrain the Gaussian wave functions and obtained a family of regular Gaussian solutions. This family corresponds to our family of regular solutions $S_n(a)$, $S_n^-(a)$ parametrized by the boundary value $S_n^-(a_1)$ [Sec. II C]. We have shown in general that all such regular solutions describe practically the same state of the Lorentzian universes (up to exponentially small corrections). Our considerations are more general than those of Ref. [10], where the existence of a family of regular solutions was demonstrated explicitly in a particular model with a negative cosmological constant.

Our analysis can also be applied to artificial situations such as those treated numerically

by Rubakov *et al.* in Ref. [2]. We may consider Eq. (23) with arbitrary functions $V(a)$, $\omega_n(a)$ as long as the potential $V(a)$ has the same qualitative behavior as the function of Eq. (6), providing a potential barrier. Our statements about the existence and the behavior of regular solutions hold for any suitably well-behaved functions $V(a)$ and $\omega_n(a)$. The unique Gaussian vacuum can be found either by a perturbative expansion or numerically.

In Ref. [2] the functions $\omega_n(a)$ and $V(a)$ were selected so that the resulting Lorentzian universe had an asymptotic region $a \to -\infty$, where $\omega_n(a) \to \omega_n^{(0)} = const$. The asymptotic in-vacuum is then well-defined and is specified by $S_n(a \to -\infty) = \omega_n^{(0)}$. However, the Gaussian vacuum for such situations will generally specify $S_n(a)$ such that

$$\lim_{a \to -\infty} [S_n(a) - \omega_n(a)] \equiv \Delta S_n \neq 0. \tag{53}$$

The quantity ΔS_n can be found using the methods of Appendix A 2. If the adiabaticity condition is satisfied, all ΔS_n are exponentially small. (Note that the potentials used in Ref. [2] do not satisfy the adiabaticity condition.)

Nonzero values of ΔS_n imply that the in-vacuum is an infinite superposition of semi-classical wave functions with different occupation numbers over the Gaussian vacuum. In fact, it is a squeezed state with squeezing parameters $\zeta_n \approx \Delta S_n / 2\omega_n^{(0)}$. Even if states with large particle numbers are exponentially suppressed to the left of the barrier $a < a_1$, critical branching may still occur. The under-barrier suppression of lower-energy branches is also exponential, and thus one is faced with a quantitative question of which of the two effects prevails. The answer to this question is likely to be model-dependent, and it is conceivable that a wave function starting in an in-vacuum state in the asymptotic region can exhibit critical branching under the barrier and be dominated by highly excited branches after the tunneling. We shall not attempt to address this issue in the present paper.

In the physically realistic case, when there is no asymptotically static region, the Gaussian vacuum can serve as a suitable definition of the vacuum state. The wave function for "tunneling from nothing" can then be obtained as a limit of tunneling from a small recollapsing universe when the energy of that universe vanishes, $\epsilon_r \to 0$. In this limit, the Gaussian vacuum becomes the de Sitter invariant Bunch-Davies vacuum state.

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Appendix A: PROPERTIES OF A REGULAR VACUUM SOLUTION

1. The regularity condition

Here we analyze the regularity condition [Eqs. (9) and (30)] and demonstrate that the regularity condition for all a is equivalent to imposing the regularity condition only at the turning point. First we show that if $|\zeta_n(t_0)| < 1$ at some $t = t_0$ within a Lorentzian region, then $|\zeta_n(t)| < 1$ for all other t. Then we demonstrate that the condition $|\zeta_n(\tau)| < 1$ will hold in the whole Euclidean region $a_1 < a < a_2$ if it holds at the turning point a_2. [For the other branch $\zeta_n^-(\tau)$, a similar argument will show that the regularity condition at $a = a_1$ is sufficient.] No assumptions of adiabaticity of $\omega_n(a)$ are made.

The function $\zeta_n(a)$ is a differentiable (complex) function satisfying a first-order differential equation with continuous coefficients, and a Cauchy problem will have a unique solution. Therefore each point $\zeta_n^{(0)}$ in the two-dimensional configuration space (the complex ζ_n plane) has a unique solution $\zeta_n(a)$ starting from $\zeta_n^{(0)}$ at some $a = a_0$.

First, consider a Lorentzian region (either $0 < a < a_1$ or $a > a_2$). The function $\zeta_n(t)$ satisfies Eq. (31) and therefore

$$Re\left[\zeta_n^* \dot{\zeta}_n\right] = \frac{1}{2}\frac{d}{dt}\left|\zeta_n\right|^2 = -\frac{\dot{\omega}_n}{2\omega_n}\left[1 - \left|\zeta_n\right|^2\right] Re\,\zeta. \tag{A1}$$

The quantity $2Re\left[\zeta_n^* \dot{\zeta}_n\right]$ is the "radial velocity" at a point ζ_n. It follows that any trajectory $\zeta_n(t)$ starting on the circle $|\zeta_n| = 1$ will remain on that circle. From uniqueness, it follows that no trajectory can cross the unit circle: any solution $\zeta_n(t)$ that is not entirely on the circle is either completely inside or completely outside of the circle.

Second, consider a Euclidean region where $\zeta_n(\tau)$ satisfies Eq. (32). A similar calculation for $|\zeta_n| = 1$ gives

$$\frac{1}{2}\frac{d}{d\tau}\left|\zeta_n\right|^2 = 2\omega_n \left|\zeta_n\right|^2 > 0 \text{ at } \left|\zeta_n\right| = 1. \tag{A2}$$

Therefore all trajectories $\zeta_n(\tau)$ that cross the unit circle must go outwards at the crossing point. The condition $|\zeta_n(\tau_2)| < 1$ guarantees $|\zeta_n(\tau)| < 1$ for all $\tau < \tau_2$. If we take τ_2 to be the value of τ at the second turning point $a = a_2$, then the required statement $|\zeta_n(a)| < 1$ for $a_1 < a < a_2$ follows.

2. An approximate solution

The main statement of this section is the following. If the adiabaticity condition

$$\left| \frac{1}{2\omega_n^2} \frac{d\omega_n}{dt} \right| \ll 1, \quad \left| \frac{1}{2\omega_n^2} \frac{d\omega_n}{d\tau} \right| \ll 1, \tag{A3}$$

is satisfied and $\omega_(t)$, $\omega_(\tau)$ are slowly-changing and non-oscillating functions, then there exists a regular solution ζ_n of Eqs. (31), (32) which is always small, $|\zeta_n| \ll 1$. The proof is constructive and uses an integral equation for ζ_n to build a perturbative expansion. The method is based on a time-dependent perturbation theory; a somewhat similar but more cumbersome treatment is in Ref. [7]. As a by-product we shall find *all* regular solutions $\zeta_n(a)$, parametrized by the boundary condition at $a = a_2$.

It will be convenient to consider one equation for ζ_n as a function of one complex variable instead of two Eqs. (31)-(32) using two different time variables. Start with Eq. (23) and define the new "time" variable θ by

$$\theta(a) \equiv \int_{a_2}^{a} \frac{\omega_n(a)}{\sqrt{\varepsilon_r - V(a)}} da. \tag{A4}$$

Here the square root has the standard branch cut, $Re\sqrt{z} > 0$, and the contour integration in Eq. (A4) uses $a \to a + i\delta$ with $\delta > 0$ at the poles $a = a_{1,2}$. The value of θ is real and positive for $a > a_2$ and coincides with the phase $\theta(t) = \int_{t_2}^{t} \omega_n dt$. In the Euclidean region ($a_1 < a < a_2$) we obtain, with the above branch cut, $\theta(\tau) = i\int_{\tau}^{\tau_2} \omega_n d\tau$. In the Lorentzian region $0 < a < a_1$ we have $\theta(t) = i\theta_b + \int_{t}^{t_1} \omega_n dt$, where

$$\theta_b \equiv \int_{\tau_1}^{\tau_2} \omega_n d\tau = \int_{a_1}^{a_2} \frac{\omega_n(a) da}{\sqrt{V(a) - \varepsilon_r}} \tag{A5}$$

is the under-barrier "Euclidean phase" and $t_{1,2}$, $\tau_{1,2}$ are the values corresponding to $a = a_{1,2}$. For our purposes it is enough to consider only the half-plane $Im\,\theta \geq 0$.

Denote for convenience

$$f(a) \equiv \frac{1}{2\omega_n^2} \frac{d\omega_n}{dt} = \frac{m^2 a \sqrt{\varepsilon_r - V(a)}}{2(n^2 + m^2 a^2)^{3/2}}. \tag{A6}$$

If the adiabaticity condition is satisfied, the value of f is everywhere small and we can find a bound f_0 such that $|f(a)| < f_0 \ll 1$.

The equation for $\zeta_n(\theta)$ is

$$\frac{d\zeta_n}{d\theta} = 2i\zeta_n - (1 - \zeta_n^2) f(\theta). \tag{A7}$$

From our analysis in Appendix A 1 it follows that a solution $\zeta_n(a)$ which is regular, $|\zeta_n(a)| < 1$ for all a, is uniquely determined by its value at the second turning point $a = a_2$ (i.e. at $\theta = 0$). Therefore we assume a boundary condition $\zeta_n(\theta = 0) = b$ with an arbitrary $|b| < 1$. Then the solution of Eq. (A7) satisfies an integral equation,

$$\zeta_n(\theta) = be^{2i\theta} - \int_0^\theta e^{2i(\theta - \theta')} \left(1 - \zeta_n^2(\theta')\right) f(\theta') d\theta'. \tag{A8}$$

The integration in Eq. (A8) is understood as contour integration along e.g. a straight line connecting 0 and θ. [The function $f(a)$ has a branch point at $a = in/m$ and we assume that an appropriate branch cut is imposed in the complex θ plane.]

We assume that $|f(\theta)| < f_0$ for the relevant values of θ, where f_0 is a fixed number. Then the integral equation for $\zeta_n(\theta)$ can be solved iteratively, starting with $\zeta_n \equiv 0$. This corresponds to a perturbation theory expansion in f_0 for Eq. (A7).

We now show that the iteration of Eq. (A8) always converges as long as $Im\, \theta > 0$. Denote $\zeta_n^{(k)}(\theta)$ the k-th element of the iteration sequence. The initial function is

$$\zeta_n^{(0)}(\theta) = be^{2i\theta} - \int_0^\theta e^{2i(\theta - \theta')} f(\theta') d\theta', \tag{A9}$$

and the next approximations are found as

$$\zeta_n^{(k+1)}(\theta) = \zeta_n^{(0)}(\theta) + \int_0^\theta e^{2i(\theta - \theta')} \left[\zeta_n^{(k)}(\theta')\right]^2 f(\theta') d\theta'. \tag{A10}$$

The difference between successive approximations is

$$\begin{aligned} \zeta_n^{(k+1)} - \zeta_n^{(k)} = & \int_0^\theta d\theta' \, f(\theta') e^{2i(\theta - \theta')} \\ & \times \left(\left[\zeta_n^{(k)}(\theta')\right]^2 - \left[\zeta_n^{(k-1)}(\theta')\right]^2 \right). \end{aligned} \tag{A11}$$

Now we shall estimate the integral in Eq. (A11) and show that the LHS tends to 0 as $k \to \infty$.

From $0 < Im\,\theta' < Im\,\theta$ we get $|\exp[2i(\theta'-\theta)]| \leq 1$. Since $|\zeta_n(\theta)| < 1$ and $|f(\theta)| \leq f_0$, we can estimate

$$\begin{aligned}
&\left|\zeta_n^{(k+1)}(\theta) - \zeta_n^{(k)}(\theta)\right| \\
&< f_0 \int_0^\theta \left|\left[\zeta_n^{(k)}(\theta')\right]^2 - \left[\zeta_n^{(k-1)}(\theta')\right]^2\right| d\theta' \\
&< 2f_0 \int_0^\theta \left|\zeta_n^{(k)}(\theta') - \zeta_n^{(k-1)}(\theta')\right| d\theta'.
\end{aligned} \tag{A12}$$

Starting from $\left|\zeta_n^{(1)}(\theta) - \zeta_n^{(0)}(\theta)\right| < f_0\,|\theta|$, we can prove by induction that

$$\left|\zeta_n^{(k)}(\theta) - \zeta_n^{(k-1)}(\theta)\right| < \frac{(2f_0\,|\theta|)^k}{2k!}. \tag{A13}$$

This sequence clearly tends to 0 as $k \to \infty$ at fixed θ.

In terms of real time variables, Eq. (A8) can be rewritten as e.g. for the S_n^- branch in the Euclidean region,

$$\begin{aligned}
\zeta_n^-(\tau) &= \zeta_n^-(\tau_1) \exp\left[-2\int_{\tau_1}^{\tau} \omega_n d\tau\right] \\
&- \int_{\tau_1}^{\tau} \exp\left[-2\int_{\tau'}^{\tau} \omega_n d\tau\right] \left(1 - \left[\zeta_n^-(\tau')\right]^2\right) f(\tau') d\tau'.
\end{aligned} \tag{A14}$$

We have obtained the solution $\zeta_n(\theta)$ of Eq. (A7) for the boundary condition $\zeta_n(\theta = 0) = b$ (with arbitrary $|b| < 1$) as the limit

$$\zeta_n(\theta) = \lim_{k\to\infty} \zeta_n^{(k)}(\theta) \tag{A15}$$

of a sequence $\zeta_n^{(k)}$ defined by Eqs. (A9)-(A10).

Now we can show that if a boundary value is small, $|b| \ll f_0$, then ζ_n is always small and at most of order f_0 in the adiabatic case $f_0 \ll 1$. The first iteration [Eq. (A9)] clearly satisfies $\left|\zeta_n^{(0)}(\theta)\right| \sim f_0$; each subsequent iteration will only add terms of higher order in f_0. Therefore, the dominant term of the solution is given by Eq. (A9). Since $Im\,\theta \geq 0$, it is clear from Eq. (A9) that $|\zeta_n(\theta)|$ is of order $|f_0|$ everywhere. We now only need to check that the oscillating integral in Eq. (A9) does not accumulate a large value when θ is large. [This could happen, for example, in the case of a parametric resonance when $\omega(t)$ is oscillating.] By assumption $f(\theta)$ is itself a slowly-changing function of θ and we can approximate $f(\theta) \approx f_1 + f_2\theta$ where f_1 and $f_2 \equiv df/d\theta$ are small constants of order f_0. Then the integral over one oscillation between θ and $\theta + 2\pi$ is

$$\int_0^{2\pi} (f_1 + f_2\theta) e^{-2i\theta} d\theta = i\pi f_2. \tag{A16}$$

The value accumulated over many oscillations between θ_1 and $\theta_2 \equiv \theta_1 + 2\pi k$ can be approximated as

$$\sum_{j=0}^{k-1} i\pi \frac{df}{d\theta}(\theta_1 + 2\pi j) \approx \frac{i}{2} \int_{\theta_1}^{\theta_2} \frac{df}{d\theta} d\theta = \frac{i}{2} \left[f(\theta_2) - f(\theta_1) \right]. \tag{A17}$$

This value is bounded by $|f_0|$. Therefore the integral of Eq. (A9) remains small also for large values of θ.

3. An asymptotic series for the adiabatic case

We can use Eq. (A8) to obtain an asymptotic expansion for $\zeta_n(\theta)$ if the adiabaticity condition $f_0 \ll 1$ holds. We shall also assume that $b \sim f_0$. The expansion is in the number of derivatives in θ as well as in powers of f and is found through integration by parts, e.g.

$$\int_0^\theta e^{2i(\theta-\theta')} f(\theta') d\theta' \sim -\sum_{n=0}^{\infty} \frac{e^{2i\theta}}{(2i)^{n+1}} \frac{d^n f(\theta)}{d\theta^n} \bigg|_0^\theta . \tag{A18}$$

Similarly to the argument leading to Eq. (A12), one can show that the iteration sequence of Eq. (A8) gives a convergent sequence of asymptotic series in which the k-th iteration changes only terms of order k and higher. Therefore the asymptotic series for the solution $\zeta_n(\theta)$ is the limit of this sequence. [Each asymptotic series in the sequence, like the series in Eq. (A18), may not actually converge.] In the Lorentzian region $0 < a < a_1$ we can omit the exponentially suppressed terms proportional to $\exp(2i\theta)$. Then the first terms of the resulting asymptotic series are

$$\zeta_n(\theta) \sim \frac{f}{2i} + \frac{f'}{(2i)^2} + \frac{f'' - f^3}{(2i)^3} + \frac{f''' - 5f'f^2}{(2i)^4} + ... \tag{A19}$$

(here the prime denotes $d/d\theta$ and all derivatives of f are evaluated at the same point θ). Converted back to the time variable t, this becomes

$$\begin{aligned} \zeta_n(t) \;\sim\; & -\frac{i}{4}\frac{\dot{\omega}_n}{\omega_n^2} - \frac{1}{8}\left(\frac{\ddot{\omega}_n}{\omega_n^3} - 2\frac{\dot{\omega}_n^2}{\omega_n^4} \right) \\ & + \frac{i}{16}\left(\frac{\dddot{\omega}_n}{\omega_n^4} - 7\frac{\dot{\omega}_n \ddot{\omega}_n}{\omega_n^5} \right) + ... \end{aligned} \tag{A20}$$

This asymptotic series is a power series in the adiabatic parameter and necessarily misses any exponentially small contributions.

4. Asymptotic series from the WKB approximation

In the adiabatic case the WKB approximation may be applied to Eq. (43) to obtain a solution

$$\nu_n(t) \propto \frac{1}{\sqrt{W}} \exp\left[-i\int^t W dt\right]. \tag{A21}$$

The auxiliary function $W(t)$ is found as an asymptotic WKB series. One may iterate the equation

$$W^{(k+1)} = \sqrt{\omega_n^2 - \frac{1}{2}\frac{\ddot{W}^{(k)}}{W^{(k)}} + \frac{3}{4}\left[\frac{\dot{W}^{(k)}}{W^{(k)}}\right]^2}, \tag{A22}$$

starting from $W^{(0)} \equiv \omega_n(\lambda t)$ and expanding in the formal adiabatic parameter λ. At the end one sets $\lambda = 1$. Then the solution $S_n(t)$ is obtained from Eqs. (42), (46) and transformed into $\zeta_n(t)$ using Eq. (29). The first terms of the resulting series are

$$S_n(t) \sim \omega_n + \frac{\dot{\omega}_n}{2i\omega} - \left(\frac{\ddot{\omega}_n}{4\omega_n^2} - \frac{3}{8}\frac{\dot{\omega}_n^2}{\omega_n^3}\right) + ..., \tag{A23}$$

$$\zeta_n(t) \sim -\frac{i}{4}\frac{\dot{\omega}_n}{\omega_n^2} - \frac{1}{8}\left(\frac{\ddot{\omega}_n}{\omega_n^3} - 2\frac{\dot{\omega}_n^2}{\omega_n^4}\right) + ... \tag{A24}$$

It is clear that Eq. (A24) should coincide with the asymptotic series obtained above in Eq. (A20) using a different approach. An asymptotic expansion in powers of the adiabatic parameter will necessarily miss any exponentially small contributions, due to the nature of the power-law expansion. The remaining asymptotic series is unique for a given function $\zeta_n(t)$, whether it was obtained from a WKB expansion or from any other procedure. An advantage of the method of Appendix A 3 is that it can compute exponentially small terms in the solution.

5. Existence and uniqueness of the Gaussian vacuum

Here we prove that for any continuous function $\omega_n(\tau)$ on an interval $[\tau_1, \tau_2]$ there exists a unique pair of (complex) functions $S_n(\tau)$, $S_n^-(\tau)$ that satisfy the equations

$$\frac{dS_n}{d\tau} = S_n^2 - \omega_n^2, \tag{A25}$$

$$-\frac{dS_n^-}{d\tau} = \left[S_n^-\right]^2 - \omega_n^2, \tag{A26}$$

the regularity conditions for any $\tau \in [\tau_1, \tau_2]$,

$$0 < Re\, S_n(\tau),\ Re\, S_n^-(\tau) < +\infty \tag{A27}$$

and the matching conditions

$$S_n^-(\tau_1) = S_n(\tau_1), \quad S_n^-(\tau_2) = S_n(\tau_2). \tag{A28}$$

Consider an auxiliary function $g(s,\tau)$ defined as the value $g(s,\tau) \equiv S_n(\tau)$ obtained by solving Eq. (A25) with the boundary condition $s = S_n(\tau_2)$. Due to the uniqueness theorem for first-order differential equations, the function $g(s,\tau)$ is differentiable and provides a one-to-one map of the complex s plane at any fixed τ. It is also clear that $g(s,\tau)$ has real values for real s. From our results in Appendix A 1 it follows that $0 < Re\, g(s,\tau) < +\infty$ for any s such that $0 \leq Re\, s < +\infty$ and for $\tau_1 \leq \tau < \tau_2$. The final useful property of $g(s,\tau)$ is $|\partial g/\partial s| < 1$ for $0 < Re\, s < +\infty$. We can prove it as follows. The function $\partial g(s,\tau)/\partial s$ as a function of τ satisfies

$$\frac{d}{d\tau}\frac{\partial g}{\partial s} = 2g(s,\tau)\frac{\partial g}{\partial s}, \quad \frac{\partial g}{\partial s}(\tau = \tau_2) = 1. \tag{A29}$$

This equation determines the function $\partial g/\partial s$ at fixed s as

$$\frac{\partial g}{\partial s}(\tau) = \exp\left[-2\int_{\tau}^{\tau_2} g(s,\tau)d\tau\right]. \tag{A30}$$

Since $Re\, g(s,\tau) > 0$, we obtain $|\partial g/\partial s| < 1$ for any $\tau \in [\tau_1, \tau_2]$. This means, in particular, that the map $s \to g(s,\tau_1)$ decreases the distance between points in the complex s plane.

Similarly, we define the function g^- by solving Eq. (A26) with the boundary condition $S_n^-(\tau_1) = s$ to find $S_n^-(\tau) \equiv g^-(s,\tau)$. The function $g^-(s,\tau)$ has the same properties as $g(s,\tau)$.

The problem of finding S_n, S_n^- is now equivalent to solving the algebraic equation $g(g^-(s,\tau_1),\tau_2) = s$ for s. The map $s \to g(g^-(s,\tau_1),\tau_2)$ clearly has the same properties as the functions g and g^-. The existence of a real positive solution $s > 0$ follows from the fact that the functions are continuous, have real values for real s, and satisfy $0 < g(s,\tau), g^-(s,\tau) < +\infty$ for $0 \leq s < +\infty$. The solution is unique because if we assume that $s_1 \neq s_2$ are two solutions, then the distance between s_1 and s_2 is decreased after applying the map $s \to g\left(g^-(s,\tau_1),\tau_2\right)$, which is a contradiction since by assumption $s_{1,2}$ are stationary points of this map. [The unique solution can be obtained numerically by iterating the map.]

6. Uniqueness of the regular solution for infinite barriers

Here we consider the equation

$$\frac{dS}{d\tau} = S^2 - \omega^2(\tau) \tag{A31}$$

on the interval $0 \le \tau < +\infty$. We call a solution $S(\tau)$ regular if $0 < Re\, S(\tau) < +\infty$ for all $\tau \ge 0$. The main statement is that Eq. (A31) has a unique regular solution $S(\tau)$ if the function $\omega^2(\tau)$ is continuous and bounded from below at large enough τ, namely $\omega(\tau) \ge f$ at $\tau > \tau_0$, with a suitable constant $f > 0$.

The technical conditions of this statement [continuity and a lower bound on $\omega(\tau)$] are sufficient but not necessary. The continuity of $\omega(\tau)$ is used only to ensure that the Cauchy problem for Eq. (A31) has a unique solution that is a continuous function of the initial condition.

It is enough to consider the case $\omega(\tau) \ge f$ for all $\tau > 0$ because a unique regular solution in a domain $\tau > \tau_0$ is uniquely extended to a regular solution for $0 \le \tau \le \tau_0$.

Firstly, we prove that any regular solution $S(\tau)$ must satisfy

$$Re\, S(\tau) \ge f \tag{A32}$$

for all $\tau \ge 0$. For this it is enough to show that a regular solution satisfies $Re\, S(\tau) > f - \varepsilon$ for arbitrary $\varepsilon > 0$. The differential equation for $R(\tau) \equiv Re\, S(\tau)$ is

$$\frac{dR}{d\tau} = R^2 - [Im\, S(\tau)]^2 - \omega^2(\tau) \le R^2 - f^2. \tag{A33}$$

So the function $R(\tau)$ cannot grow faster than a solution $R_0(\tau)$ of the equation

$$\frac{dR_0}{d\tau} = R_0^2 - f^2. \tag{A34}$$

Given a regular solution $S(\tau)$ and a point τ_1, we can choose $R_0(\tau)$ such that $R_0(\tau_1) \le Re\, S(\tau_1)$ and then it follows that $R_0(\tau)$ is a lower bound for $Re\, S(\tau)$ for $0 \le \tau < \tau_1$. Since we know that $Re\, S(\tau) > 0$, we can use the boundary condition $R_0(\tau_1) = 0$ at any $\tau_1 > 0$ to obtain lower bounds on $Re\, S(\tau)$. The general solution of Eq. (A34) is

$$R_0(\tau) = f \frac{1 - Ae^{2f\tau}}{1 + Ae^{2f\tau}}, \tag{A35}$$

where A is an integration constant. It is easy to see that any solution $R_0(\tau)$ equal to zero at some large τ must be exponentially close to f at smaller τ. More precisely, for any ε such

that $0 < \varepsilon < f$ and for any τ_1 the solution $R_0(\tau)$ will satisfy $R_0(\tau) > f - \varepsilon$ for $0 \leq \tau < \tau_1$ if we impose the boundary condition $R_0(\tau_2) = 0$ at

$$\tau_2 = \tau_1 + \frac{1}{2f} \ln \frac{2f - \varepsilon}{\varepsilon}. \tag{A36}$$

But the range of τ is infinite and we can chose τ_1 to be arbitrarily large. Since $R_0(\tau)$ is a lower bound for $Re\, S(\tau)$, it follows that $Re\, S(\tau) > f - \varepsilon$ for $0 \leq \tau < \tau_1$ with any $\tau_1 > 0$. Therefore, we have shown that any regular solution of Eq. (A31) satisfies Eq. (A32) for all τ.

Secondly, we prove that the relative difference between any two regular solutions $S_1(\tau)$, $S_2(\tau)$ decreases exponentially with diminishing τ. More precisely, for any two values τ_a, τ_b such that $\tau_a < \tau_b$,

$$\frac{|S_1(\tau_a) - S_2(\tau_a)|^2}{|S_1(\tau_a)|^2 + |S_2(\tau_a)|^2} \leq \frac{|S_1(\tau_b) - S_2(\tau_b)|^2}{|S_1(\tau_b)|^2 + |S_2(\tau_b)|^2} e^{-2f(\tau_b - \tau_a)}, \tag{A37}$$

as long as these solutions satisfy $Re\, S_{1,2}(\tau) \geq f$ for $\tau_a \leq \tau \leq \tau_b$. This can be proved directly by using Eqs. (A31)-(A32) to evaluate

$$\begin{aligned} &\frac{d}{d\tau} \ln \frac{|S_1(\tau) - S_2(\tau)|^2}{|S_1(\tau)|^2 + |S_2(\tau)|^2} \\ =\ & 2\frac{|S_1|^2\, Re\, S_2 + |S_2|^2\, Re\, S_1 + \omega^2 Re(S_1 + S_2)}{|S_1(\tau)|^2 + |S_2(\tau)|^2} \\ \geq\ & 2f. \end{aligned} \tag{A38}$$

From Eqs. (A32) and (A37) we find that the difference between any two regular solutions at $\tau = 0$ must be equal to zero. This follows from Eq. (A37) with $\tau_a = 0$: the relative difference by definition cannot exceed 1, and the RHS can be made arbitrarily small by choosing large enough τ_b. Therefore the regular solution is unique.

The existence of a regular solution can be proved by construction. We find a regular solution $S(\tau)$ as the limit of a sequence of solutions that are regular for a finite part of the interval $0 \leq \tau < +\infty$. Consider a (real) function $S(\tau; \tau_0)$ defined as the solution of Eq. (A31) with the boundary condition $S(\tau_0; \tau_0) = 0$. The function $S(\tau; \tau_0)$ is not a regular solution because it satisfies $Re\, S(\tau) > 0$ only on the interval $0 \leq \tau < \tau_0$ but not at larger τ. We now show that the limit of $S(\tau; \tau_0)$ as $\tau_0 \to +\infty$ is a regular solution. The existence of the pointwise limit (taken separately at each τ)

$$S(\tau) \equiv \lim_{\tau_0 \to +\infty} S(\tau; \tau_0) \tag{A39}$$

follows from Eq. (A37): the relative difference between functions becomes exponentially small when τ_0 grows. It is clear that the resulting function $S(\tau)$ will satisfy Eq. (A32). It remains to show that the function $S(\tau)$ defined by Eq. (A39) is actually a solution of Eq. (A31). This follows from a continuity argument. A solution of Eq. (A31) is a continuous function of an initial condition. Therefore the limit of Eq. (A39) is the same as the solution of Eq. (A31) with the boundary condition

$$S(\tau = 0) = \lim_{\tau_0 \to +\infty} S(\tau = 0; \tau_0). \tag{A40}$$

This completes the proof of the existence and uniqueness of the regular solution of Eq. (A31).

Appendix B: APPLICABILITY OF THE APPROXIMATIONS

In this Appendix we analyze the applicability of the Gaussian and WKB approximations to the Wheeler-DeWitt equation. We use the WKB approximation in two places in our calculation. First, the WKB approximation is used in substituting the Gaussian ansatz into the Wheeler-DeWitt equation, when all terms of order $O(\hbar)$ are ignored. Second, the WKB approximation is applied to Eq. (43). We analyze the latter approximation first.

1. Using the WKB approximation for Eq. (43)

The WKB ansatz of Eq. (A21) is a good approximation for Eq. (43) as long as

$$\dot{\omega}_n \ll \omega_n^2. \tag{B1}$$

Comparing this with Eq. (A20), we find that the squeezing parameter $\zeta_n(a)$ is small if and only if the WKB approximation is valid.

Consider the behavior of $\zeta_n(a)$ in the Euclidean region where we use the time variable τ (overdots will denote derivatives by τ). Taking the leading term of Eq. (A20),

$$\zeta_n(a) \approx \frac{\dot{\omega}_n}{4\omega_n^2} = \frac{m^2 a \sqrt{V(a) - \varepsilon}}{4(n^2 + m^2 a^2)^{3/2}} \tag{B2}$$

and estimating $V(a) - \varepsilon \lesssim a^2$, we obtain

$$\zeta_n(a) < \frac{m^2 a^2}{4(n^2 + m^2 a^2)^{3/2}}. \tag{B3}$$

At fixed n, this function reaches a maximum at $ma = n\sqrt{2}$, which gives a bound

$$\zeta_n(a) < \frac{1}{2n\sqrt{27}}. \tag{B4}$$

Therefore $|\zeta_n(a)| \ll 1$ and the WKB approximation is valid if $n \gg 1$. Similarly, one can show that the WKB is valid if $ma \gg n$ or if $ma \ll n$. The only case when the WKB may not apply to Eq. (B2) is when $n \sim 1$ and $ma \sim 1$ at the same time; the maximum value of ζ_n is at most ≈ 0.1 in this case.

In the Lorentzian region, Eq. (B2) gives at large a

$$|\zeta_n| \sim \frac{H}{4m}. \tag{B5}$$

The adiabaticity condition is satisfied at large a only if $H \ll m$. However, we know from Appendix A 1 that the solution $\zeta_n(a)$ remains regular in Lorentzian regions even if the adiabadicity condition does not hold somewhere in those regions.

2. Using the WKB approximation for the WDW equation

When we substitute the Gaussian ansatz into the Wheeler-DeWitt equation [Eq. (5)], we disregard terms of order $O(\hbar)$. These terms are $\hbar S_0''$, $\hbar S_n''$, and the "backreaction" term $\hbar \sum_n S_n$. The WKB approximation is valid if the disregarded terms are smaller than the typical magnitude of other terms in the respective equation. For the first two terms, the conditions are

$$\hbar S_0'' \ll V(a), \quad \hbar S_n'' \ll \omega_n^2. \tag{B6}$$

The WKB approximation may only be valid away from the turning points. We find

$$\left|\frac{S_0''}{V}\right| = \frac{|V'|}{2V^{3/2}} \approx H^2 \frac{|1-2\lambda^2|}{\lambda^2(1-\lambda^2)^{3/2}}; \tag{B7}$$

this is small when $H \ll 1$ away from turning points. We can also verify that

$$S_n'' = \frac{d}{da}\left[\frac{2m^2 a}{\omega_n(a)}\right] = \frac{n^2 m}{\omega_n^3} \ll \omega_n^2 \tag{B8}$$

for $ma \gg n$ or $ma \ll n$.

Now we consider the backreaction of the excited field modes on the metric. The divergent sum $\hbar \sum_n S_n$ corresponds to the infinite zero-point energy of the oscillators χ_n. We assume that this infinity is absorbed by an appropriate renormalization [9]. Any finite terms

remaining after renormalization will be of order of the squared curvature $\sim H^4 \ll 1$ and we neglect them here (see e.g. Ref. [16]). The remaining correction due to backreaction consists of replacing the effective potential $V(a)$ by

$$V(a) + \Delta V = V(a) - 2\hbar \sum_{n=0}^{\infty} n^3 p_n, \tag{B9}$$

where p_n are occupation numbers in the modes χ_n relative to an appropriate vacuum and we have inserted the multiplicity factor n^3 [see Ref. [6], Eq. (37) for details]. The average occupation number in the state of the mode χ_n characterized by $\zeta_n(a)$, relative to a vacuum described by $\zeta_n^{(0)}(a)$, is found from Eq. (52),

$$\langle p_n \rangle = |\beta_n|^2 = \frac{\left|\zeta_n - \zeta_n^{(0)}\right|^2}{(1 - |\zeta_n|^2)(1 - \left|\zeta_n^{(0)}\right|^2)}. \tag{B10}$$

We shall not underestimate $\langle p_n \rangle$ here if we take $\zeta_n^{(0)} \equiv 0$, which corresponds to the vacuum of the instantaneous diagonalization. The sum over large n can be estimated using the WKB approximation for ζ_n [cf. Eq. (A20)] applicable at large n (as shown in Appendix B 1). The leading term is

$$|\zeta_n| \approx \left| \frac{\dot{\omega}_n}{4\omega_n^2} \right|. \tag{B11}$$

In this limit ζ_n is small and we can take $1 - |\zeta_n|^2 \approx 1$. At fixed a the squeezing parameter $\zeta_n(a)$ decays as n^{-3} at large n [Eq. (B2)]. This decay is fast enough so that the sum over modes in Eq. (B9) converges. We obtain

$$\begin{aligned} \frac{\Delta V}{V} &\sim \frac{1}{V} \sum_{n=0}^{\infty} \frac{2n^3 |\zeta_n|^2}{1 - |\zeta_n|^2} \\ &\approx \sum_{n=0}^{\infty} \frac{n^3 m^4 a^2}{8(n^2 + m^2 a^2)^3} \approx \frac{m^2}{32}. \end{aligned} \tag{B12}$$

Since $m \ll 1$ in Planck units, we find that fractional change in $V(a)$ due to backreaction is small. [The occupation numbers are computed here in the instantaneous diagonalization vacuum where ζ_n is the instantaneous squeezing parameter. The particle numbers in a higher-order adiabatic vacuum are expected to be even smaller.]

3. Validity of the Gaussian approximation

In using the Gaussian ansatz, we disregard the terms which are quartic in χ_n but retain the terms quadratic in χ_n. The Gaussian ansatz gives rise to a quartic term

$$\frac{\chi_n^4}{4}\left(\frac{dS_n}{da}\right)^2. \tag{B13}$$

This term can be disregarded at small enough χ_n if the following inequality holds,

$$\frac{\chi_n^4}{4}\left(\frac{dS_n}{da}\right)^2 \ll \chi_n^2\omega_n^2. \tag{B14}$$

The latter condition gives a corridor around the line $\chi_n = 0$ in which the Gaussian ansatz is valid, $|\chi_n| < \chi_{\max}$. We can show that the width of this corridor is never small. The width of the corridor (divided by a, since χ_n is rescaled by a) can be estimated using

$$S_n(a) \approx \omega_n + \frac{\dot{\omega}_n}{2\omega_n} \tag{B15}$$

and Eq. (A9). We find

$$\left(\frac{\chi_{\max}}{a}\right)^2 \sim \frac{2\omega_n(a)}{dS_n/da} = \frac{2\omega_n\sqrt{V(a)}}{a(S_n^2-\omega_n^2)} = 1 + \frac{n^2}{m^2a^2} > 1. \tag{B16}$$

This expression is bounded from below and therefore the allowed corridor never shrinks to zero width.

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